IDEAL THEORY AND PRÜFER DOMAINS

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Fractional Ideals

Throughout this section, R is an integral domain.

Definition 1. A *fractional ideal* J of an integral domain R is an R-submodule of $\mathrm{qf}(R)$ for which there exists a nonzero $r \in R$ such that rJ is an ideal of R.

So fractional ideals of R are subsets of $\mathrm{qf}(R)$ of the form $\frac{1}{r}I$, where $r \in R$ and I is an ideal of R. A fractional ideal J is called *principal* if there exists $x \in \mathrm{qf}(R)$ such that $J = Rx$. It is clear that every ideal (resp., principal ideal) of an integral domain is a fractional ideal (resp., principal fractional ideal). Conversely, if a fractional ideal (resp., principal fractional ideal) of R is contained in R, then it is an ideal (resp., principal ideal).

Proposition 2. *For an integral domain R, the following statements hold.*

(1) *Every finitely generated R-submodule of* $\mathrm{qf}(R)$ *is a fractional ideal.*
(2) *If R is Noetherian, then every fractional ideal is finitely generated.*

Proof. (1) Let J be a finitely generated R-submodule of $\mathrm{qf}(R)$, and take $q_1, \dots, q_n \in J$ such that $J = Rq_1 + \cdots + Rq_n$. For each $i \in [\![1, n]\!]$, we can write $q_i = r_i/s_i$ for some $r_i, s_i \in R$ with $s_i \neq 0$. After setting $s = s_1 \cdots s_n$, we see that $sq_1, \dots, sq_n \in R$. As a result, $sJ = Rsq_1 + \cdots + Rsq_n$ is an ideal of R. Hence J is a fractional ideal.

(2) Now suppose that R is Noetherian, and let J be a fractional ideal of R. Then rJ is an ideal of R for some nonzero $r \in R$ and, because R is Noetherian, we can write $rJ = Ra_1 + \cdots + Ra_k$ for some $a_1, \dots, a_k \in R$. Hence the equality $J = Ra_1/r + \cdots + Ra_k/r$ holds, and so J is finitely generated. □

We can define the sum, product, and quotient (or colon) of two fractional ideals in the same way it is done for ideals and, in this case, we obtain fractional ideals.

Proposition 3. *Let R be an integral domain. Then the following statements hold for any fractional ideals J_1 and J_2 of R.*

(1) $J_1 + J_2$ *is a fractional ideal.*
(2) $J_1 \cap J_2$ *is a fractional ideal.*
(3) $J_1J_2 := \big\{ \sum_{i=1}^n a_ib_i \mid n \in \mathbb{N},\ a_1, \dots, a_n \in J_1,\ \text{and } b_1, \dots, b_n \in J_2 \big\}$ *is a fractional ideal.*

(4) $(J_1 : J_2) = \{q \in \mathrm{qf}(R) : qJ_2 \subseteq J_1\}$ *is a fractional ideal if* J_2 *is nonzero.*

Proof. (1) We know $J_1 + J_2$ is an R-submodule of $\mathrm{qf}(R)$. In addition, if $r_1J_1 \subseteq R$ and $r_2J_2 \subseteq R$ for some nonzero $r_1, r_2 \in R$, then $r_1r_2(J_1 + J_2) = r_2(r_1J_1) + r_1(r_2J_2) \subseteq R$.

(2) The intersection $J_1 \cap J_2$ is clearly an R-submodule of $\mathrm{qf}(R)$. Also, if $r \in R$ is a nonzero element such that $rJ_1 \subseteq R$, then $r(J_1 \cap J_2) \subseteq rJ_1 \subseteq R$.

(3) From the given definition, one can readily see that J_1J_2 is an R-submodule of $\mathrm{qf}(R)$. In addition, if $r_1, r_2 \in R$ are nonzero elements such that $r_1J_1 \subseteq R$ and $r_2J_2 \subseteq R$, then $r_1r_2\sum_{i=1}^n a_ib_i = \sum_{i=1}^n (r_1a_i)(r_2b_i) \in R$ for any $n \in \mathbb{N}$, $a_1, \dots, a_n \in J_1$, and $b_1, \dots, b_n \in J_2$. Hence $r_1r_2J_1J_2 \subseteq R$.

(4) It is routine to check that $(J_1 : J_2)$ is an R-submodule of $\mathrm{qf}(R)$. Take nonzero elements $r_1, r_2 \in R$ such that $r_1J_1 \subseteq R$ and $r_2J_2 \subseteq R$. Fix a nonzero $d \in J_2$, and set $r := r_1r_2d$. Then $r \neq 0$ and $r \in r_1(r_2J_2) \subseteq r_1R \subseteq R$. In addition, for any $q \in (J_1 : J_2)$, the fact that $qJ_2 \subseteq J_1$ implies that $rq = r_1r_2(qd) \in r_2r_1J_1 \subseteq r_2R \subseteq R$. Hence $r(J_1 : J_2) \subseteq R$. □

Since multiplication of fractional ideals is clearly associative, it follows from Proposition 3 that the set $\mathscr{F}(R)$ of nonzero fractional ideals of R is a commutative semigroup under multiplication with identity element R.

Definition 4. A nonzero fractional ideal of R is called *invertible* if it is invertible as an element of the semigroup $\mathscr{F}(R)$.

So if J is an invertible fractional ideal of R, then there is only one inverse of J in $\mathscr{F}(R)$, an it is not hard to verify that this inverse is $(R : J)$. We let $\mathscr{I}(R)$ denote the set of invertible elements of $\mathscr{F}(R)$. Clearly, $\mathscr{I}(R)$ is a subgroup of $\mathscr{F}(R)$. It is convenient to let J^{-1} denote the fractional ideal $(R : J)$ even when J is not invertible, and we do so. If J is a nonzero principal fractional ideal and $q \in \mathrm{qf}(R)$ satisfies $J = qR$, then it follows immediately that $J^{-1} = q^{-1}R$, and so $J^{-1}J = R$. Thus, every nonzero principal fractional ideal is invertible, and so the set $\mathrm{Prin}(R)$ consisting of all nonzero principal fractional ideals of R is a subgroup of $\mathscr{I}(R)$. Putting all together we obtain the following proposition.

Proposition 5. *If* R *is an integral domain, then* $\mathscr{I}(R)$ *is an abelian group, and* $\mathrm{Prin}(R)$ *is a subgroup of* $\mathscr{I}(R)$.

As the following example illustrates, not every finitely generated fractional ideal of an integral domain R is invertible, even when $\dim R = 1$.

Example 6. Consider the ring $R = F[x,y]/(y^2 - x^3)$, where F is a field. The assignments $x \mapsto t^2$ and $y \mapsto t^3$ determine a ring isomorphism $R \cong F[t^2, t^3]$. Identify R with $F[t^2, t^3]$, and consider the ideal $I = Rt^2 + Rt^3$. Then $(R : I) = t^{-1}(R + Rt)$ and, therefore, $I(R : I) = Rt + Rt^2 + Rt^3 \subseteq Rt$. As a result, I is a finitely generated ideal that is not invertible. Finally, observe that $\dim R = 1$ because the extension $F[t^2, t^3] \subseteq F[t]$ is integral.

Invertible ideals, on the other hand, are finitely generated.

Proposition 7. *For an integral domain R, the following statements hold.*

1. *Every invertible (fractional) ideal of R is finitely generated.*
2. *If R is local, then every invertible (fractional) ideal is principal.*

Proof. (1) Let I be an invertible (fractional) ideal of R. Take J to be the fractional ideal satisfying $IJ = R$, and write $1 = \sum_{i=1}^{n} a_i b_i$ for $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in J$. Then for every $x \in I$, we see that $x = \sum_{i=1}^{n} a_i(xb_i)$. Since $xb_i \in R$ for every $i \in [\![1, n]\!]$, it follows that $x \in Ra_1 + \dots + Ra_n$. So $I \subseteq Ra_1 + \dots + Ra_n$. Since the reverse inclusion also holds, I is a finitely generated ideal.

(2) Let R be a local ring with maximal ideal M. Let I be an invertible (fractional) ideal of R with inverse J. As in the previous part, we can write $1 = \sum_{i=1}^{n} a_i b_i$ for $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in J$. As $1 \notin M$, we see that $a_j b_j \notin M$ for some $j \in [\![1, n]\!]$. Since R is local, $a_j b_j \in R^\times$. Then for every $x \in I$, we obtain that $x = u(xb_j)a_j \in Ra_j$, where $u := (a_j b_j)^{-1} \in R$. Hence $I \subseteq Ra_j$. Since the reverse inclusion clearly holds, I is a principal ideal. □

Therefore we have the following diagram of implications, where F.I. stands for fractional ideal and f.g. for finitely generated.

$$\textbf{Principal F.I.} \; \underset{\Longrightarrow}{\overset{\text{local}}{\Longleftarrow}} \; \textbf{Invertible F.I.} \; \underset{\Longrightarrow}{\not\Longleftarrow} \; \textbf{f.g. F.I.} \; \underset{\Longrightarrow}{\overset{\text{Noetherian}}{\Longleftarrow}} \; \textbf{F.I.}$$

Recall that an R-module is projective if it is a direct summand of a free R-module. We have seen before that an R-module is projective if and only if there exists a free R-module F and R-module homomorphisms $\alpha \colon F \to M$ and $\beta \colon M \to F$ such that $\alpha \circ \beta = 1_M$. We conclude this lecture characterizing invertible ideals in terms of projective modules.

Theorem 8. *Let R be an integral domain. Then a nonzero fractional ideal of R is invertible if and only if it is a projective R-module.*

Proof. For the direct implication, suppose that J is an invertible fractional ideal. Write $1 = \sum_{i=1}^{n} x_i y_i$ for $x_1, \dots, x_n \in J$ and $y_1, \dots, y_n \in J^{-1}$. Let F be a free R-module with basis elements $m_1, \dots, m_n$, and let $\alpha \colon F \to J$ be the R-module homomorphism induced by the assignments $m_i \mapsto x_i$ (for every $i \in [\![1, n]\!]$). One can easily verify that the map $\beta \colon J \to F$ defined by $\beta(x) = \sum_{i=1}^{n} (xy_i) m_i$ is an R-module homomorphism. Now we see that

$$(\alpha \circ \beta)(x) = \alpha\Big(\sum_{i=1}^{n} (xy_i) m_i \Big) = \sum_{i=1}^{n} (xy_i) x_i = x$$

for every $x \in J$. Hence $\alpha \circ \beta = 1_J$, and so J is a projective R-module.

For the reverse implication, suppose that J is a nonzero fractional ideal of R, which is a projective R-module. Then there exist a free R-module F and R-module homomorphisms $\alpha\colon F \to J$ and $\beta\colon J \to F$ such that $\alpha \circ \beta = 1_J$. Let S be a free generating set of F. Now let r be a nonzero element of J, and write $\beta(r) = \sum_{i=1}^n r_i m_i$, where $r_1, \dots, r_n \in R$ and $m_1, \dots, m_n$ are distinct elements in S. Set $a_i = \alpha(m_i)$ and $q_i = r_i/r \in \text{qf}(R)$ for every $i \in [\![1, n]\!]$. For each $x \in J$, we can write $\beta(x) = \sum_{i=1}^n x_i m_i + \sum_{m \in T} c_m m$, where $T := S \setminus \{m_1, \dots, m_n\}$ and $x_1, \dots, x_n, c_m \in R$ for each $m \in T$ (here $c_m = 0$ for all but finitely many $m \in T$). After considering coefficients in

$$\sum_{i=1}^n (x r_i) m_i = x\beta(r) = r\beta(x) = \sum_{i=1}^n (r x_i) m_i + \sum_{m \in T} (r c_m) m,$$

we can easily see that $rc_m = 0$ for all $m \in T$, and so that $q_i x = (r_i/r)x = x_i \in R$ for every $i \in [\![1, n]\!]$. Hence $q_i \in J^{-1}$ for each $i \in [\![1, n]\!]$. Since $r \neq 0$, from

$$r = (\alpha \circ \beta)(r) = \alpha\Big(\sum_{i=1}^n r_i m_i\Big) = \sum_{i=1}^n r_i \alpha(m_i) = \sum_{i=1}^n a_i r_i = r\Big(\sum_{i=1}^n a_i q_i\Big)$$

we obtain that $\sum_{i=1}^n a_i q_i = 1$, which implies that $JJ^{-1} = R$. Hence one can conclude that J is invertible. ☐

Exercise

Exercise 1. *Let $R = \mathbb{Z}[\sqrt{-3}] := \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$, and consider the fractional ideal $J := R + R\omega$, where ω is the primitive cube root of unity $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$.*

(1) *Compute J^{-1} and JJ^{-1}.*
(2) *Is J an invertible fractional ideal of R?*

Exercise 2. *Let R be an integral domain, let S be a multiplicative subset of R, and let I is an ideal of R. Prove that if I is invertible in R, then $S^{-1}I$ is invertible in $S^{-1}R$.*

Exercise 3. *Let R be an integral domain having finitely many maximal ideals. Prove that every invertible fractional ideal of R is principal.*

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